

DECOMPOSITION OF WAVELET REPRESENTATIONS AND MARTIN BOUNDARIES

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ABSTRACT. We study a decomposition problem for a class of unitary representations associated with wavelet analysis, wavelet representations, but our framework is wider and has applications to multi-scale expansions arising in dynamical systems theory for non-invertible endomorphisms.

Our main results offer a direct integral decomposition for the general wavelet representation, and we solve a question posed by Judith Packer. This entails a direct integral decomposition of the general wavelet representation. We further give a detailed analysis of the measures contributing to the decomposition into irreducible representations. We prove results for associated Martin boundaries, relevant for the understanding of wavelet filters and induced random walks, as well as classes of harmonic functions.

Our setting entails representations built from certain finite-to-one endomorphisms r in compact metric spaces X , and we study their dilations to automorphisms in induced solenoids. Our wavelet representations are covariant systems formed from the dilated automorphisms. They depend on assigned measures μ on X . It is known that when the data (X, r, μ) are given the associated wavelet representation is typically reducible. By introducing wavelet filters associated to (X, r) we build random walks in X , path-space measures, harmonic functions, and an associated Martin boundary.

We construct measures on the solenoid (X_∞, r_∞) , built from (X, r) . We show that r_∞ induces unitary operators U on Hilbert space \mathcal{H} and representations π of the algebra $L^\infty(X)$ such that the pair (U, r_∞) , together with the corresponding representation π forms a crossed-product in the sense of C^* -algebras. We note that the traditional wavelet representations fall within this wider framework of (\mathcal{H}, U, π) covariant crossed products.

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1. INTRODUCTION

We study a decomposition problem for a class of unitary representations associated with wavelet analysis, even though our framework is wider and has applications outside multi-scale wavelet

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expansions, see details below. One powerful tool in the construction of families of multi-scale wavelets (see e.g., [Dau92, Mal98, Jor06]) is an introduction of a finite system of filters. Here we understand the notion of “filter” in the sense of signal processing. In this context, each filter will be a function of a complex number z (a frequency variable), and for many purposes it is enough to consider only a phase of z , so we may restrict attention to the case when points z are in the 1-torus. Each function $m_i(z)$, $i = 0, 1, \dots, N - 1$ typically supports a frequency band. With these conventions, in wavelet considerations, the function m_0 represents a low-pass filter, i.e., passing low frequency signals. When a suitable Fourier expansion is introduced for the filter functions we arrive at the masking coefficients that determine some particular wavelet. This framework includes both traditional wavelet systems in the Hilbert space $L^2(\mathbb{R}^d)$ in some number of dimensions d , as well as orthonormal wavelet bases on fractals, as studied by two of the present authors, see [DJ07, DJ06, BK10].

Analysis of filters. Continuing with filter functions m_0 on the circle group \mathbb{T} , we consider, for every fixed z in \mathbb{T} , then the absolute squared m_0 with some normalization $W(z) = |m_0(z)|^2/N$. With this we then get a family of probability distributions on the set of N solutions in \mathbb{T} to the equation $w^N = z$ (see (2.1)–(2.5) below). There will be a solution w in each of the N frequency-bands. In the special case when $N = 2$, the function m_0 , or the system of functions m_0 and m_1 , are called a quadrature mirror filter (QMF). The reason for this is that two other operations, down-sampling, and up-sampling, allow one to build a discrete wavelet algorithm with dual filters, the dual one is the “mirror”.

Here we have adopted a more general framework: Instead of \mathbb{T} we will consider a compact metric space X , and our filter functions will be functions from X into the complex plane. We further generalize the choice of endomorphism $r(z) = z^N$, considering here instead an endomorphism r in X which is onto, and for which each pre-image of points in X is a finite subset; so finite-to-one endomorphisms. In both the traditional wavelet case, and in the more general framework, we end up with dynamical systems in a solenoid.

When the pair (X, r) is given as specified, there is a standard way of building a solenoid $X_\infty = X_\infty(r)$ over X (see (2.7) and (2.8) below). There are several advantages working with the solenoid:

- (i) The endomorphism r in X induces an automorphism r_∞ in X_∞ (see (2.8)).
- (ii) The transition probabilities W on X induce a random walk on the solenoid which encodes properties of the representations induced by the prescribed wavelet filters [DJ07]; these representations are known as *wavelet representations* in the literature.
- (iii) With this random walk we are able to compute transition probabilities, harmonic functions, and associated Martin boundaries.
- (iv) Points in the solenoid may be thought of as random-walk paths; for each point x in X , we will have an infinite random-walk path, represented as a subset of the solenoid.
- (v) Fixing the function $W = |m_0(\cdot)|^2/\#r^{-1}(r(\cdot))$ on X we get a path space measure P . We will be interested in the family of measures (P_x) , x in X , with P_x conditioned on the set of paths starting at x (see (2.12)).

To specify wavelet representations we must then also have a prescribed measure μ on X (see Definition 2.1). For the theory of wavelet representations, see for example [HL08, Lar07b, Lar07a, Pac04, Pac08b, Pac08a]. For an early treatment of transfer operators R_W in multi-scale wavelets, see [Jor01]. The relevance of the R_W harmonic functions (i.e., solutions h to $R_W h = h$) in the decomposition theory for the corresponding wavelet representation ρ_W was pointed out there.

Specifically, [Jor01] has the idea of computing operators in the commutant of ρ_W from R_W -harmonic functions.

For computations of R_W -harmonic functions of concrete wavelets, see also [BJ02]. The theory of R_W harmonic functions was developed recently by a number of other authors; in [Dut06] is introduced the study of intertwining operators for pairs of wavelet representations.

In our present wider context, the R_W -harmonic functions enter in equation (2.5) below, and they underlie our considerations in Section 4 regarding Martin boundary.

In our present setting, in an earlier paper [DS] a necessary condition was given on the data (X, r, μ) for when the associated wavelet representation is reducible. For the reader's convenience, we have stated it as Theorem 2.4 below.

Summary of results. Our main result here (Theorem 3.3) offers a direct integral decomposition for the general wavelet representation. This completely solves a question posed by Professor Judith Packer, see e.g., [Pac08a, BFMP10, BLP⁺10, BLM⁺09, BFMP09]. In our Theorem 3.3 we offer a direct integral decomposition of the general wavelet representation, and our Theorem 4.5 deals with a derivation of the measures contributing to the decomposition. Our results yield as clean a decomposition for the general wavelet representation into irreducibles as is realistically feasible.

In Section 4, we have included a result on an associated Martin boundary. Even though it is not used directly, it is certainly relevant for the understanding of our random-walk harmonic functions, and the question about the Martin boundary naturally presents itself. In fact, we obtained our result in Section 4 in response to a question asked of us by Erin Pearse.

Indeed, there are some intriguing connections to random-walk models studied recently in papers by one of the present authors and E. Pearse (see e.g., [JP10] and [DJ10]). These are computations for infinite weighted graphs G . In both instances we get transition probabilities and associated random walks.

If G is a graph as in [JP10], the condition for the context of these studies is that, for every vertex in G , there are only a finite number of transitions possible to neighboring vertices. But the Markov processes in [JP10] are reversible, and therefore the associated boundaries are more amenable. By contrast, our transition processes are non-reversible, except for some trivial special cases. More specifically, the reason our random walks are not reversible, is that transition happens from one point x to one of the distinct subnodes y (neighbors) where y will be one of the finite number of solutions to the equation $r(y) = x$. Hence these transitions never return, unless we consider cases when x might be a periodic point, in which case they might return.

2. MEASURES ON THE SOLENOID

Before turning to our direct integral result, we begin with some preliminaries regarding measures on solenoids. Since our starting point is a given finite-to-one endomorphism r in a compact metric space X , it is then natural to look for a way of corresponding to this a unitary operator U in a Hilbert space \mathcal{H} , such that U together with (X, r) satisfy a covariance relation; see (i) in Theorem 2.2 below. The introduction of suitable measures on the associated solenoid (X_∞, r_∞) , built from (X, r) , then gets us a representation π of the algebra $L^\infty(X)$ such that U , together with r_∞ , form a crossed-product in the sense of C^* -algebras. This is possible since r_∞ is an automorphism. We will refer to a crossed-product system (\mathcal{H}, U, π) as a wavelet representation.

Indeed, in [DJ07], we proved that the traditional wavelet representations fall within this wider framework of (\mathcal{H}, U, π) covariant crossed products. Specifically, in the special case when $X = \mathbb{T}$,

and the endomorphism r is just the power mapping $r(z) = z^N$ (for a fixed integer $N > 1$), then it can be seen that a covariant crossed products indeed specializes to a unitary representation of a corresponding N -Baumslag-Solitar group; see e.g., [DJ08, Dut06]. Even in the case of these classical Baumslag-Solitar groups, our understanding of the unitary representations and their decompositions is so far only partial.

Definition 2.1. Let X be a compact metric space and $r : X \rightarrow X$ be a finite-to-one, onto, Borel measurable map. Let μ be a *strongly invariant* Borel probability measure on X , i.e.

$$(2.1) \quad \int f d\mu = \int \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} f(y) d\mu(x),$$

for any bounded Borel function on X .

A function m_0 on X is called a *quadrature mirror filter (QMF)* if

$$(2.2) \quad \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} |m_0(y)|^2 = 1, \quad (x \in X)$$

In what follows we will assume that:

$$(2.3) \quad \text{the set of zeroes for } m_0 \text{ has } \mu\text{-measure zero.}$$

Given a QMF m_0 we define

$$(2.4) \quad W(x) = \frac{|m_0(x)|^2}{\#r^{-1}(r(x))}, \quad (x \in X).$$

Then the function W satisfies the following equation:

$$(2.5) \quad \sum_{r(y)=x} W(y) = 1, \quad (x \in X).$$

Equation (2.5) can be interpreted as an assignment of transition probabilities: the probability of transition from x to $y \in r^{-1}(x)$ is equal to $W(y)$.

A function h on X is called R_W -harmonic if

$$(2.6) \quad \sum_{r(y)=x} W(y)h(y) = h(x), \quad (x \in X).$$

Theorem 2.2. [DJ07] *There exists a Hilbert space \mathcal{H} , a unitary operator U on \mathcal{H} , a representation π of $L^\infty(X)$ on \mathcal{H} and an element φ of \mathcal{H} such that*

- (i) (Covariance) $U\pi(f)U^{-1} = \pi(f \circ r)$ for all $f \in L^\infty(X)$.
- (ii) (Scaling equation) $U\varphi = \pi(m_0)\varphi$
- (iii) (Orthogonality) $\langle \pi(f)\varphi, \varphi \rangle = \int f d\mu$ for all $f \in L^\infty(X)$.
- (iv) (Density) $\{U^{-n}\pi(f)\varphi \mid n \in \mathbb{N}, f \in L^\infty(X)\}$ is dense in \mathcal{H} .

Moreover they are unique up to isomorphism.

Definition 2.3. We call the system $(\mathcal{H}, U, \pi, \varphi)$ in Theorem 2.2, the *wavelet representation* associated to the function m_0 .

We recall some facts from [DJ07]. The wavelet representation can be realized on a solenoid as follows. Let

$$(2.7) \quad X_\infty := \left\{ (x_0, x_1, \dots) \in X^\mathbb{N} \mid r(x_{n+1}) = x_n \text{ for all } n \geq 0 \right\}.$$

We call X_∞ the *solenoid* associated to the map r .

On X_∞ consider the σ -algebra generated by cylinder sets. Let $r_\infty : X_\infty \rightarrow X_\infty$

$$(2.8) \quad r_\infty(x_0, x_1, \dots) = (r(x_0), x_0, x_1, \dots) \text{ for all } (x_0, x_1, \dots) \in X_\infty.$$

Then r_∞ is a measurable automorphism on X_∞ .

Define $\theta_0 : X_\infty \rightarrow X$,

$$(2.9) \quad \theta_0(x_0, x_1, \dots) = x_0.$$

The following commutative diagram summarizes the relation between the maps r, r_∞, θ_0 :

$$\begin{array}{ccc} X_\infty & \xrightarrow{r_\infty} & X_\infty \\ \theta_0 \downarrow & & \downarrow \theta_0 \\ X & \xrightarrow{r} & X \end{array} \quad , \quad \theta_0 \circ r_\infty = r \circ \theta_0$$

Define for $m \geq 0$ the projection $\theta_m : X_\infty \rightarrow X$,

$$\theta_m(x_0, x_1, \dots) = x_m.$$

The measure μ_∞ on X_∞ will be defined by constructing some path measures P_x on the fibers $\Omega_x := \{(x_0, x_1, \dots) \in X_\infty \mid x_0 = x\}$.

On Ω_{x_0} we will consider the infinite product topology which is defined by the basis of open sets: for $n \geq 0$, $x_1, \dots, x_n \in X$ with $r(x_{j+1}) = x_j$, $j \in \{0, \dots, n-1\}$,

$$(2.10) \quad V_{x_0, \dots, x_n} := \{(z_0, z_1, \dots) \in \Omega_{x_0} : z_0 = x_0, \dots, z_n = x_n\}.$$

With this topology Ω_{x_0} is a compact Hausdorff space.

Let

$$c(x) := \#r^{-1}(r(x)), \quad W(x) = |m_0(x)|^2/c(x), \quad (x \in X).$$

Then

$$(2.11) \quad \sum_{r(y)=x} W(y) = 1, \quad (x \in X)$$

$W(y)$ can be thought of as the transition probability from $x = r(y)$ to one of its roots y .

For $x \in X$, the path measure P_x on Ω_x is defined on cylinder sets by

$$(2.12) \quad P_x(\{(x_n)_{n \geq 0} \in \Omega_x \mid x_1 = z_1, \dots, x_n = z_n\}) = W(z_1) \dots W(z_n)$$

for any $z_1, \dots, z_n \in X$.

This value can be interpreted as the probability of the random walk to go from x to z_n through the points x_1, \dots, x_n .

Next, define the measure μ_∞ on X_∞ by

$$(2.13) \quad \int f d\mu_\infty = \int_X \int_{\Omega_x} f(x, x_1, \dots) dP_x(x, x_1, \dots) d\mu(x)$$

for bounded measurable functions on X_∞ .

Consider now the Hilbert space $\mathcal{H} = L^2(\mu_\infty)$. Define the operator

$$(2.14) \quad U\xi = (m_0 \circ \theta_0)\xi \circ r_\infty, \quad (\xi \in L^2(X_\infty, \mu_\infty)).$$

Define the representation of $L^\infty(X)$ on \mathcal{H}

$$(2.15) \quad \pi(f)\xi = (f \circ \theta_0)\xi, \quad (f \in L^\infty(X), \xi \in L^2(X_\infty, \mu_\infty)).$$

Let $\varphi = 1$ be the constant function 1 on X_∞ .

Theorem 2.4. [DJ07] *Suppose m_0 is non-singular, i.e., $\mu(\{x \in X \mid m_0(x) = 0\}) = 0$. Then the data $(\mathcal{H}, U, \pi, \varphi)$ forms the wavelet representation associated to m_0 .*

Theorem 2.5. [DS] *Suppose $r : (X, \mu) \rightarrow (X, \mu)$ is ergodic. Assume $|m_0|$ is not constant 1 μ -a.e., non-singular, i.e., $\mu(m_0(x) = 0) = 0$, and $\log |m_0|^2$ is in $L^1(X)$. Then the wavelet representation $(\mathcal{H}, U, \pi, \varphi)$ is reducible.*

We will be interested in the decomposition of the wavelet representation into irreducibles. We need a few more notations and lemmas.

Definition 2.6. Define

$$\begin{aligned} \tilde{m}_0 &= 1, \quad \tilde{m}_n = (m_0 \circ \theta_0) \cdot (m_0 \circ \theta_0 \circ r_\infty) \dots (m_0 \circ \theta_0 \circ r_\infty^{n-1}), \text{ for } n \geq 1, \\ \tilde{m}_n &= \frac{1}{(m_0 \circ \theta_0 \circ r_\infty^{-1}) \dots (m_0 \circ \theta_0 \circ r_\infty^n)}, \text{ for } n < 0. \end{aligned}$$

The function $\tilde{m} : X_\infty \times \mathbb{Z} \rightarrow \mathbb{C}^*$ defined by $\tilde{m}(x, n) = \tilde{m}_n(x)$ gives a one-cocycle for the action of \mathbb{Z} on X_∞ determined by r_∞ .

The fact that U is an isometry implies the following lemma.

Lemma 2.7. *For $\xi \in L^2(X_\infty, \mu_\infty)$:*

$$\int \xi d\mu_\infty = \int |\tilde{m}_n|^2 \xi \circ r_\infty^n d\mu_\infty, \quad (n \in \mathbb{Z}).$$

3. THE DECOMPOSITION OF THE WAVELET REPRESENTATION

Definition 3.1. We say that a subset \mathcal{F} of X_∞ is a *fundamental domain* if, up to μ_∞ -measure zero:

$$\bigcup_{n \in \mathbb{Z}} r_\infty^n(\mathcal{F}) = X_\infty \quad \text{and} \quad r_\infty^n(\mathcal{F}) \cap r_\infty^m(\mathcal{F}) = \emptyset \text{ for } n \neq m.$$

Definition 3.2. For $z = (z_0, z_1, \dots)$ in X_∞ define the following representation: consider the Hilbert space

$$\mathcal{H}_z := \left\{ (\xi_n)_{n \in \mathbb{Z}} : \sum_{n \in \mathbb{Z}} |\xi_n|^2 |\tilde{m}_n(z)|^2 < \infty \right\},$$

with inner product

$$\langle \xi, \eta \rangle_{\mathcal{H}_z} := \sum_{n \in \mathbb{Z}} \xi_n \bar{\eta}_n |\tilde{m}_n(z)|^2.$$

Note that we avoid here the points $z \in X_\infty$ such that one of the functions $\tilde{m}_n(z) = 0$. Since m_0 is non-singular, such points form a set of μ_∞ -measure zero.

Define the unitary operator

$$U_z(\xi_n)_{n \in \mathbb{Z}} = (m_0 \circ \theta_0 \circ r_\infty^n(z) \xi_{n+1})_{n \in \mathbb{Z}}.$$

Define the representation of π of $L^\infty(X)$:

$$\pi_z(f)(\xi_n)_{n \in \mathbb{Z}} = (f \circ \theta_0 \circ r_\infty^n(z) \xi_n)_{n \in \mathbb{Z}}, \quad (f \in L^\infty(X)).$$

The representation π_z is defined for bounded functions on X , not just essentially bounded. The μ -measure zero sets will affect the individual representations π_z but not their direct integral (see below).

Theorem 3.3. *In the hypotheses of Theorem 2.5, there exist a fundamental domain \mathcal{F} . The wavelet representation associated to m_0 has the following direct integral decomposition:*

$$[\mathcal{H}, U, \pi] = \int_{\mathcal{F}}^{\oplus} [\mathcal{H}_z, U_z, \pi_z] d\mu_\infty(z),$$

where the component representations $[\mathcal{H}_z, U_z, \pi_z]$ in the decomposition are irreducible for a.e., z in \mathcal{F} , relative to μ_∞ .

Proof. We state the irreducibility of the component representations in a lemma:

Lemma 3.4. *For μ_∞ almost every $z \in X_\infty$, the objects $[\mathcal{H}_z, U_z, \pi_z]$ form an irreducible representation.*

Proof. One has to check that U_z is unitary, π_z is a representation and $U_z \pi_z(f) U_z^{-1} = \pi_z(f \circ r)$ for all $f \in L^\infty(X)$. All these follow from simple computations.

To see that the representation is irreducible for μ_∞ -a.e. z , take z to be non-periodic, i.e., $r_\infty^n(z) \neq z$ for all $n \neq 0$. Then $\{\pi_z(f) : f \in L^\infty(X)\}$ forms a maximal abelian subalgebra with cyclic vector δ_0 (see [Tak02, Corollary III.1.3]), where $\delta_0(n) = 1$ for $n = 0$, and $\delta_0(n) = 0$ otherwise. Then, an operator A that commutes with U_z and π_z has to be of the form $\pi_z(g)$ for some $g \in L^\infty(X)$. Since A commutes with U_z we have $\pi_z(g \circ r) = U_z \pi_z(g) U_z^{-1} = \pi_z(g)$. This implies that g is constant on $\{r_\infty^n(z) : n \in \mathbb{Z}\}$, so A is a multiple of the identity. \square

We begin the proof as in the proof of the main result in [DS].

From the QMF relation and the strong invariance of μ we have

$$\int_X |m_0|^2 d\mu = \int_X \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} |m_0(y)|^2 d\mu = 1.$$

By Jensen's inequality we have

$$a := \int_X \log |m_0|^2 d\mu \leq \log \int_X |m_0|^2 d\mu = 0.$$

Since \log is strictly concave, and $|m_0|^2$ is not constant μ -a.e., it follows that the inequality is strict, and $a < 0$.

Since r is ergodic, applying Birkoff's ergodic theorem, we obtain that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |m_0 \circ r^k|^2 = \int_X \log |m_0|^2 d\mu = a, \quad \mu - \text{a.e.}$$

This implies that

$$\lim_{n \rightarrow \infty} (|m_0(x)m_0(r(x)) \dots m_0(r^{n-1}(x))|^2)^{1/n} = e^a < 1, \mu - \text{a.e.}$$

Take b with $e^a < b < 1$.

By Egorov's theorem, there exists a measurable set A_0 , with $\mu(A_0) > 0$, such that

$$(|m_0(x)m_0(r(x)) \dots m_0(r^{n-1}(x))|^2)^{1/n}$$

converges uniformly to e^a on A_0 . (Taking A_0 smaller if needed we can assume $\mu(A_0) < 1$.) This implies that there exists an n_0 such that for all $m \geq n_0$:

$$(|m_0(x)m_0(r(x)) \dots m_0(r^{m-1}(x))|^2)^{1/m} \leq b \text{ for } x \in A_0$$

and so

$$(3.1) \quad |m_0(x)m_0(r(x)) \dots m_0(r^{m-1}(x))|^2 \leq b^m, \text{ for } m \geq n_0 \text{ and all } x \in A_0.$$

Next, given $m \in \mathbb{N}$, we compute the probability of a sequence $(z_n)_{n \in \mathbb{N}} \in X_\infty$ to have $z_m \in A_0$. We have, using the strong invariance of μ :

$$\begin{aligned} P(z_m \in A_0) &= \mu_\infty(\{(z_n)_n \mid z_m \in A_0\}) = \int_{X_\infty} \chi_{A_0} \circ \theta_m d\mu_\infty \\ &= \int_X \frac{1}{\#r^{-m}(z_0)} \sum_{r(z_1)=z_0, \dots, r(z_m)=z_{m-1}} |m_0(z_1)|^2 \dots |m_0(z_m)|^2 \chi_{A_0}(z_m) d\mu(z_0) \\ &= \int_X |m_0(z_m)m_0(r(z_m)) \dots m_0(r^{m-1}(z_m))|^2 \chi_{A_0}(z_m) d\mu(z_m) \\ &= \int_X |m_0(x)m_0(r(x)) \dots m_0(r^{m-1}(x))|^2 \chi_{A_0}(x) d\mu(x). \end{aligned}$$

Then

$$\sum_{m=1}^{\infty} P(z_m \in A_0) = \sum_{m \geq 1} \int_X |m_0(x)m_0(r(x)) \dots m_0(r^{m-1}(x))|^2 \chi_{A_0} d\mu(x) < \infty$$

and we used (3.1) in the last inequality.

Now we can use Borel-Cantelli's lemma, to conclude that the probability that $z_m \in A_0$ infinitely often is zero. Thus, for μ_∞ -a.e. $z := (z_n)_n$, there exists k_z (depending on the point) such that $z_n \notin A_0$ for $n \geq k_z$. In other words, if $B_0 = X \setminus A_0$ then for μ_∞ -a.e. $(z_n)_n$ in X_∞ there exists k_z such that $z_n \in B_0$ for all $n \geq k_z$.

Define now the set

$$A_\infty := \{(z_0, z_1, \dots) \in X_\infty : z_0, z_1, \dots \in B_0\}.$$

It is clear that if $(z_0, z_1, \dots) \in A_\infty$ then $r_\infty^{-1}(z_0, z_1, \dots) = (z_1, z_2, \dots)$ is in A_∞ too. Therefore $r_\infty^{-1}(A_\infty) \subseteq A_\infty$. This means also that $A_\infty \subseteq r_\infty(A_\infty)$.

From the statements above we see that for μ_∞ -a.e. (z_0, z_1, \dots) in X_∞ there exists n such that z_n, z_{n+1}, \dots are in B_0 which means that $r_\infty^{-n}(z_0, z_1, \dots)$ is in A_∞ and so $(z_0, z_1, \dots) \in r_\infty^n(A_\infty)$. Thus, up to measure zero:

$$\bigcup_{n \in \mathbb{Z}} r_\infty^n(A_\infty) = X_\infty.$$

We claim that also, up to measure zero, one has

$$\bigcap_{n \in \mathbb{Z}} r_\infty^n(A_\infty) = \emptyset.$$

Suppose (z_0, z_1, \dots) is in all $r_\infty^{-n}(A_\infty)$ for $n \geq 0$. Then $(r^n(z_0), r^{n-1}(z_0), \dots) \in A_\infty$ so $r^n(z_0) \in B_0$ for all $n \geq 0$. Since $0 < \mu(B_0) < 1$ this contradicts the fact that r is ergodic on X .

Now take $\mathcal{F} := r_\infty(A_\infty) \setminus A_\infty$. The properties of A_∞ easily imply that \mathcal{F} is a fundamental domain.

Next we check the direct integral decomposition.

Define $\Psi : L^2(X_\infty, \mu_\infty) \rightarrow \int_{\mathcal{F}}^{\oplus} \mathcal{H}_z d\mu_\infty(z)$,

$$(\Psi\xi)(z) = (\xi \circ r_\infty^n(z))_{n \in \mathbb{Z}}, \quad (\xi \in L^2(X_\infty, \mu_\infty), z \in \mathcal{F}).$$

We check that Ψ is an isometry. We use Lemma 2.7:

$$\begin{aligned} \|\xi\|^2 &= \sum_{n \in \mathbb{Z}} \int |\xi|^2 \chi_{r_\infty^n(\mathcal{F})} d\mu_\infty = \sum_{n \in \mathbb{Z}} \int |\tilde{m}_n|^2 |\xi \circ r_\infty^n|^2 \chi_{\mathcal{F}} d\mu_\infty = \int_{\mathcal{F}} \sum_{n \in \mathbb{Z}} |\tilde{m}_n|^2 |\xi \circ r_\infty^n|^2 d\mu_\infty \\ &= \int_{\mathcal{F}} \|(\Psi\xi)(z)\|_{\mathcal{H}_z}^2 d\mu_\infty(z) = \|\Psi\xi\|^2. \end{aligned}$$

To check that Ψ is onto, we can compute the inverse $(\Psi^{-1}(\xi(\cdot)_n)_{n \in \mathbb{Z}})(z) = \xi_n(r_\infty^{-n}z)$ if $z \in r_\infty^n(\mathcal{F})$. Some direct computations show that Ψ intertwines the U -operators and the representations π . \square

4. MARTIN BOUNDARY

The idea of associating to wavelet constructions a transfer operator R_W and associated harmonic functions was pioneered by W. Lawton in the two papers [Law90, Law91].

The idea is that wavelets are determined by a system of numbers, often called masking coefficients. It is possible to turn these into coefficients in filter functions m_i , and by selecting $i = 0$ (see eq (2.5)) we get transition probabilities and a transfer operator R_W ,

$$R_W f(x) = \sum_{r(y)=x} W(y) f(y), \quad (x \in X).$$

Hence R_W is determined by the prescribed masking coefficients, and the question is how properties of the masking coefficients (and therefore of R_W) determine the wavelets. It turns out that this is decided by the spectrum of R_W , including the eigenspace for eigenvalue 1, which produces the harmonic functions.

As shown in [DLS], operators in the commutant of the wavelet representation correspond to bounded R_W -harmonic functions. If we restrict such harmonic functions to inverse orbits of points we get harmonic functions for the random walk, or what we call below p -harmonic functions. The Martin boundary theory offers a way to construct such harmonic functions by means of integrals on a certain boundary. We perform these computations here to see what the p -harmonic functions are in this case.

Definition 4.1. A point $x_0 \in X$ is called *regular* if the following two conditions are satisfied:

- (i) The sets $r^{-n}(x_0)$, $n \in \mathbb{N}$ are mutually disjoint.
- (ii) None of the sets $r^{-n}(x_0)$, $n \geq 0$ intersect the set of zeroes of W .

Note that condition (i) means that x_0 is not periodic for the map r , i.e., $r^n(x_0) \neq x_0$ for any $n \geq 1$.

For a point $x_0 \in X$, define the set $\mathcal{T}(x_0) := \cup_{n \geq 0} r^{-n}(x_0)$. We call this the *tree with root at x_0* . If x_0 is regular and $x \in \mathcal{T}(x_0)$, define $n(x_0)$ to be the unique non-negative integer such that $r^{n(x_0)}(x) = x_0$.

Let $x_0 \in X$ be regular. We define now a random walk on the set $\mathcal{T}(x_0)$ and we construct its Martin boundary by following [Saw97].

For $x, y \in \mathcal{T}(x_0)$ define the transition probabilities $p(x, y)$ as follows:

$$(4.1) \quad p(x, y) := \begin{cases} W(y), & \text{if } r(y) = x, \\ 0, & \text{otherwise.} \end{cases}$$

A function u on $\mathcal{T}(x_0)$ is called *p-harmonic* if

$$(4.2) \quad u(x) = \sum_{y \in \mathcal{T}(x_0)} p(x, y) u(y), \quad (x \in \mathcal{T}(x_0)).$$

The function $p_n(x, y)$ is the n -th matrix power of $p(x, y)$ and represents the probability of transition from x to y in n steps. It can be easily seen that $p_0(x, y) = \delta_{xy}$ and

$$(4.3) \quad p_n(x, y) = \begin{cases} W(y)W(r(y)) \dots W(r^{n-1}(y)), & \text{if } r^n(y) = x, \\ 0, & \text{otherwise.} \end{cases}$$

The *Green function* or *potential function* is defined by

$$(4.4) \quad g(x, y) := \sum_{n=0}^{\infty} p_n(x, y), \quad (x, y \in \mathcal{T}(x_0)).$$

Note that, for our random walk, only one term in the sum in (4.4) is non-zero.

The *Martin kernel* is defined by

$$(4.5) \quad K(x, y) := \frac{g(x, y)}{g(x_0, y)}, \quad (x, y \in \mathcal{T}(x_0)).$$

The denominator in (4.5) is non-zero because each vertex y can be reached from x_0 eventually.

The function $K(x, \cdot)$ is bounded by some constant C_x (which we will describe below). Set

$$(4.6) \quad \rho(x, y) = \sum_{q \in \mathcal{T}(x_0)} D(q) \frac{|K(q, x) - K(q, y)| + |\delta_{qx} - \delta_{qy}|}{C_q + 1}, \quad (x, y \in \mathcal{T}(x_0)),$$

where $D(q) > 0$ for all $q \in \mathcal{T}(x_0)$ and $\sum_{q \in \mathcal{T}(x_0)} D(q) < \infty$. Here $\delta_{xy} = 1$ if $x = y$ and $\delta_{xy} = 0$ otherwise.

The *Martin compactification* $[\widehat{\mathcal{T}}(x_0), \widehat{\rho}]$ is the completion of $\mathcal{T}(x_0)$ with the metric ρ . The *Martin boundary* is defined as $\partial\mathcal{T}(x_0) := \widehat{\mathcal{T}}(x_0) \setminus \mathcal{T}(x_0)$.

As shown in [Saw97] a sequence $\{y_n\}$ in $\mathcal{T}(x_0)$ is Cauchy with respect to the metric ρ if and only if either (i) $y_n = y$ for all $n \geq n_0$ for some $y \in \mathcal{T}(x_0)$ and some $n_0 \in \mathbb{N}$, or else (ii) $\lim_{n \rightarrow \infty} y_n = \infty$ and $\lim_{n \rightarrow \infty} K(x, y_n)$ exists for all $x \in \mathcal{T}(x_0)$. (Here $\lim_{n \rightarrow \infty} y_n = \infty$ means that y_n leaves eventually any finite set and never returns.)

Thus the Martin boundary $\partial\mathcal{T}(x_0)$ is the set of equivalence classes of Cauchy sequences that satisfy the condition (ii) above.

The maps $K(x, \cdot)$, $x \in \mathcal{T}(x_0)$ extend uniquely to continuous maps on $\widehat{\mathcal{T}}(x_0)$ and we use the same notation $K(x, \cdot)$ for their extensions.

Theorem 4.2. [Martin representation theorem] *For any p -harmonic function $u(x) \geq 0$ there exists a measure ν on $\partial\mathcal{T}(x_0)$ such that*

$$(4.7) \quad u(x) = \int_{\partial\mathcal{T}(x_0)} K(x, \alpha) d\nu(\alpha), \quad (x \in \mathcal{T}(x_0)).$$

Proposition 4.3. *With the definitions above we have:*

(i) *The Green function satisfies the equation*

$$(4.8) \quad g(x, y) = \begin{cases} W(y)W(r(y)) \dots W(r^{n-1}(y)), & \text{if } r^n(y) = x \text{ for some } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(If $n = 0$ the product is defined to be 1.)

(ii) *The Martin kernel is*

$$(4.9) \quad K(x, y) = \begin{cases} \frac{1}{W(x)W(r(x)) \dots W(r^{n(x)-1}(x))} =: C_x, & \text{if } r^n(y) = x \text{ for some } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $K(x, \cdot)$ is constant C_x on the subtree $\mathcal{T}(x)$ with root at x and 0 everywhere else. Using the notation of Definition 2.6 we have that $K(x, y) = \frac{1}{W_n(x)}$, if $r^n(y) = x$; if $\#r^{-1}(x)$ is constant, then the two functions \tilde{W}_n and \tilde{m}_n differ by a multiplicative constant (see (2.4)).

(iii) *A function u on $\mathcal{T}(x_0)$ is p -harmonic if and only if*

$$(4.10) \quad u(x) = \sum_{r(y)=x} W(y)u(y), \quad (x \in \mathcal{T}(x_0)).$$

Proof. (i) follows directly from (4.3). Note that, because x_0 is regular, the number n such that $r^n(y) = x$ is unique. For (ii), if $r^n(y) \neq x$ for all n then $g(x, y) = 0$ so $K(x, y) = 0$. If $r^n(y) = x$, then we have

$$\begin{aligned} K(x, y) &= \frac{g(x, y)}{g(x_0, y)} = \frac{W(y) \dots W(r^{n-1}(y))}{W(y) \dots W(r^{n(y)-1}(y))} = \frac{1}{W(r^n(y)) \dots W(r^{n(y)-1}(y))} \\ &= \frac{1}{W(x) \dots W(r^{n(x)-1}(x))}. \end{aligned}$$

(iii) is obtained from the following computation:

$$u(x) = \sum_y p(x, y)u(y) = \sum_{r(y)=x} p(x, y)u(y) = \sum_{r(y)=x} W(y)u(y).$$

□

Theorem 4.4. *Let x_0 be a regular point in X . Define the map $\Phi : \Omega_{x_0} \rightarrow \partial\mathcal{T}(x_0)$ by*

$$(4.11) \quad \Phi(x_0, x_1, \dots) := \{x_n\},$$

i.e., to each sequence in Ω_{x_0} we associate the equivalence class of this sequence in $\partial\mathcal{T}(x_0)$.

Then Φ is a bijective homeomorphism from Ω_{x_0} onto $\partial\mathcal{T}(x_0)$.

For $x \in \mathcal{T}(x_0)$ and $(x_0, x_1, \dots) \in \Omega_{x_0}$

$$(4.12) \quad K(x, \Phi(x_0, x_1, \dots)) = \begin{cases} \frac{1}{W(x_1)W(x_2) \dots W(x_n)}, & \text{if } x = x_n \text{ for some } n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. First, we show that Φ is well defined, so $\{x_n\}$ is a sequence with the property that $\lim x_n = \infty$ and $\lim K(x, x_n)$ exists for all $x \in \mathcal{T}(x_0)$.

Since x_0 is regular, the sets $r^n(x_0)$ are disjoint, therefore any finite subset of $\mathcal{T}(x_0)$ lies in a finite union $\cup_{j \leq J} r^{-j}(x_0)$, and since $x_n \in r^{-n}(x_0)$ for all n , it follows that x_n eventually leaves this finite set and never returns.

For the second condition, take $x \in \mathcal{T}(x_0)$. Recall that $n(x)$ is the unique number such that $x \in r^{-n(x)}(x_0)$. We have two possibilities: $x_{n(x)} = x$ or not. In the first case we have that x_n is in the subtree $\mathcal{T}(x)$ for all $n \geq n(x)$ so $K(x, x_n)$ is constant C_x , by Proposition 4.3. In the second case, we have that x_n is not in the subtree $\mathcal{T}(x)$ for all $n \geq n(x)$, so $K(x, x_n)$ is constant 0. In both cases $\lim K(x, x_n)$ exists, so Φ is well defined.

Next, we check that Φ is onto. Take a sequence $\{y_n\}$ in $\mathcal{T}(x_0)$ with $\lim y_n = \infty$ and such that $\lim K(x, y_n)$ exists for all $x \in \mathcal{T}(x_0)$. Since $K(x, y)$ is either $C_x > 0$ or 0 depending on whether y is in the subtree $\mathcal{T}(x)$ or not, it follows that for all $x \in \mathcal{T}(x_0)$, either the sequence y_n is eventually contained in the subtree $\mathcal{T}(x)$ or it is eventually contained in the complement of $\mathcal{T}(x)$; it cannot jump back and forth between $\mathcal{T}(x)$ and its complement.

We will construct by induction the sequence $(x_n)_n$ in Ω_{x_0} with $\Phi(x_0, x_1, \dots) = \{y_n\}$. The first element x_0 is given. Next consider the points z_1, \dots, z_J in $r^{-1}(x_0)$. Since $\lim y_n = \infty$, eventually the sequence will be in the union of the subtrees $\cup_{j=1}^J \mathcal{T}(z_j)$. With the previous remark, one of the sets $\mathcal{T}(z_j)$ will contain the entire sequence y_n eventually. We define x_1 to be the point z_j with this property.

Inductively, if x_m has been defined such that the entire sequence y_n is in the subtree $\mathcal{T}(x_m)$ eventually, we take the points in $r^{-1}(x_m)$; since $\lim y_n = \infty$, the entire sequence will lie in $\cup_{z \in r^{-1}(x_m)} \mathcal{T}(z)$ eventually. Since y_n cannot jump back and forth between a subtree and its complement, there is one of the elements $z \in r^{-1}(x_m)$ such that the sequence y_n lies in the subtree $\mathcal{T}(z)$ eventually. We call this point x_{m+1} .

To prove that $\Phi(x_0, x_1, \dots) = \{y_n\}$ we just have to show that the sequences $\{x_n\}$ and $\{y_n\}$ are equivalent, i.e., $\lim \rho(x_n, y_n) = 0$. Take $\epsilon > 0$. There exists a finite subset F of $\mathcal{T}(x_0)$ such that $\sum_{q \notin F} 2D(q) < \epsilon$. For each $q \in F$, either y_n is in the subtree $\mathcal{T}(y_n)$ eventually, or it is in the complement of $\mathcal{T}(y_n)$ eventually. From the definition of $\{x_n\}$ we see that x_n will have exactly the same property. Thus $K(q, x_n) = K(q, y_n)$ for n large enough. Also, since x_n and y_n go to infinity, it follows that $\delta_{qx_n} = \delta_{qy_n}$ for n large enough. Therefore, for n large, the terms in the sum in (4.6) for $\rho(x_n, y_n)$ that correspond to $q \in F$ are all zero, the rest are bounded by $2 \sum_{q \notin F} D(q) < \epsilon$. So $\rho(x_n, y_n) < \epsilon$ for n large, and therefore $\Phi(x_0, x_1, \dots) = \{y_n\}$ and Φ is onto.

To see that Φ is one-to-one, take $(x_n) \neq (x'_n)$ in Ω_{x_0} . Let $n_0 \geq 1$ such that $x_{n_0} \neq x'_{n_0}$. Then for $n \geq n_0$, $K(x_{n_0}, x_n) = C_{x_{n_0}}$ and $K(x_{n_0}, x'_n) = 0$. Therefore

$$\rho(x_n, x'_n) \geq \frac{|K(x_{n_0}, x_n) - K(x_{n_0}, x'_n)|}{C_{x_{n_0}} + 1} = \frac{C_{x_{n_0}}}{C_{x_{n_0}} + 1}.$$

This implies that $\Phi(x_n) \neq \Phi(x'_n)$ so Φ is one-to-one.

To prove that Φ is continuous, take $(x_n) \in \Omega_{x_0}$ and $\epsilon > 0$. Take a finite subset F of $\mathcal{T}(x_0)$ such that $2 \sum_{q \notin F} D(q) < \epsilon$. Take n_0 such that F is contained in $\cup_{n \leq n_0} r^{-n}(x_0)$. Take $(x'_n) \in V_{x_0, \dots, x_{n_0}}$ so $x'_0 = x_0, \dots, x'_{n_0} = x_{n_0}$. Then the sequences x_n and x'_n are in the subtree $\mathcal{T}(x_{n_0})$ for $n \geq n_0$. This implies that for $q \in F$ and $n > n_0$, we have that either both x_n and x'_n lies in the subtree

$\mathcal{T}(q)$ or they both lie outside $\mathcal{T}(q)$; also $\delta_{qx_n} = \delta_{qx'_n} = 0$. Then $\rho(x_n, x'_n) \leq \sum_{q \notin F} 2D(q) < \epsilon$ so $\widehat{\rho}(\Phi(x_n), \Phi(x'_n)) \leq \epsilon$. This shows that Φ is continuous.

Since both spaces Ω_{x_0} and $\partial\mathcal{T}(x_0)$ are compact Hausdorff, it follows that Φ is a homeomorphism.

Next, we check (4.12). Take $x \in \mathcal{T}(x_0)$ and $(x_0, x_1, \dots) \in \Omega_{x_0}$. We have $K(x, \Phi(x_0, x_1, \dots)) = \lim K(x, x_n)$. If $x \neq x_n$ for all $n \geq 0$, then x_n is not in the subtree $\mathcal{T}(x)$ for all n , so $K(x, x_n) = 0$ for all n and therefore $K(x, \Phi(x_0, x_1, \dots)) = 0$.

If $x = x_n$ for some n , then for $m \geq n$, x_m is in the subtree $\mathcal{T}(x)$ so

$$K(x, x_m) = K(x_n, x_m) = \frac{1}{W(x_n)W(r(x_n)) \dots W(r^{n(x_n)-1}(x_n))} = \frac{1}{W(x_n)W(x_{n-1}) \dots W(x_1)}.$$

This proves (4.12). \square

Theorem 4.5. *For any p -harmonic function $u \geq 0$, there exists a measure ν on Ω_{x_0} such that*

$$(4.13) \quad u(x) = \frac{1}{W(x)W(r(x)) \dots W(r^{n(x)-1}(x))} \nu(V_{r^{n(x)}(x), r^{n(x)-1}(x), \dots, x}), \quad (x \in \mathcal{T}(x_0)).$$

Proof. By Theorem 4.2, there exists a measure $\hat{\nu}$ on $\partial\mathcal{T}(x_0)$ such that

$$u(x) = \int_{\partial\mathcal{T}(x_0)} K(x, \alpha) d\hat{\nu}(\alpha), \quad (x \in \mathcal{T}(x_0)).$$

Define the measure ν on Ω_{x_0} by $\nu = \hat{\nu} \circ \Phi$. Then

$$u(x) = \int_{\Omega_{x_0}} K(x, \Phi(x_0, x_1, \dots)) d\nu(x_0, x_1, \dots).$$

But $K(x, \Phi(\cdot))$ is a multiple C_x of the characteristic function of $V_{r^{n(x)}(x), r^{n(x)-1}(x), \dots, x}$. From this, (4.13) follows immediately. \square

Definition 4.6. A non-negative function ν on $\mathcal{T}(x_0)$ is called *additive* if

$$(4.14) \quad \nu(x) = \sum_{r(y)=x} \nu(y), \quad (x \in \mathcal{T}(x_0)).$$

Denote by

$$W^{(n)}(x) = W(x) \dots W(r^{n-1}(x)).$$

Corollary 4.7. *For any p -harmonic function $u \geq 0$ there exists a unique additive function ν such that*

$$(4.15) \quad u(x) = \frac{1}{W^{(n(x))}(x)} \nu(x), \quad (x \in \mathcal{T}(x_0)).$$

Conversely, if u is given by (4.15) for some additive function ν , then u is p -harmonic.

Proof. The existence can be obtained from Theorem 4.5 by defining the additive function

$$\nu(x) := \nu(V_{r^{n(x)}(x), r^{n(x)-1}(x), \dots, x}), \quad (x \in \mathcal{T}(x_0)).$$

Uniqueness is clear since $W \neq 0$ on $\mathcal{T}(x_0)$.

For the converse, we compute

$$\sum_{r(y)=x} W(y)u(y) = \sum_{r(y)=x} W(y) \frac{1}{W(y)W(r(y)) \dots W(r^{n(y)-1}(y))} \nu(y)$$

$$= \frac{1}{W(x) \dots W(r^{n(x)-1}(x))} \sum_{r(y)=x} \nu(y) = u(x).$$

□

Remark 4.8. Note that the function $\nu_0(x) = W^{(n(x))}(x)$, $x \in \mathcal{T}(x_0)$ is an additive function. Indeed

$$\begin{aligned} \sum_{r(y)=x} \nu_0(y) &= \sum_{r(y)=x} W^{(n(y))}(y) = \sum_{r(y)=x} W(y)W(r(y)) \dots W(r^{n(y)-1}(y)) \\ &= W(x) \dots W(r^{n(x)-1}(x)) \sum_{r(y)=x} W(y) = \nu_0(x) \cdot 1. \end{aligned}$$

Therefore we have the following corollary.

Corollary 4.9. *Every non-negative harmonic function is the quotient of two additive functions.*

Definition 4.10. A function U on $\mathcal{T}(x_0)$ is called a *QMF-weight* if $U \geq 0$ and

$$\sum_{r(y)=x} U(y) = 1, \quad (x \in \mathcal{T}(x_0)).$$

Proposition 4.11. *There exists a one-to-one correspondence between positive additive functions and positive QMF-weights on $\mathcal{T}(x_0)$.*

For every positive additive function ν on $\mathcal{T}(x_0)$ the function

$$U_\nu(x) = \frac{\nu(x)}{\nu(r(x))}, \quad (x \in \mathcal{T}(x_0) \setminus \{x_0\}),$$

is a QMF-weight.

Conversely, for every positive QMF-weight U , the function

$$\nu(x) = \nu(x_0)U(x)U(r(x)) \dots U(r^{n(x)-1}(x)), \quad (x \in \mathcal{T}(x_0) \setminus \{x_0\}),$$

is additive, where $\nu(x_0)$ is some fixed non-negative constant.

Proof. We have

$$\sum_{r(y)=x} U_\nu(y) = \sum_{r(y)=x} \frac{\nu(y)}{\nu(r(y))} = \frac{1}{\nu(x)} \sum_{r(y)=x} \nu(y) = 1.$$

The converse follows as in Remark 4.8. □

Definition 4.12. Let X_r be the set of points $x_0 \in X$ such that $r^n(x_0)$ is regular for all $n \geq 0$.

Remark 4.13. Note that X_r is invariant for both r and r^{-1} .

Proposition 4.14. *Let h be an R_W -harmonic function on X_r , i.e., equation (2.6) is satisfied for $x \in X_r$. Then for each $x_0 \in X_r$ there exists a unique additive function ν_{x_0} on $\mathcal{T}(x_0)$ such that*

$$(4.16) \quad h(x) = \frac{\nu_{x_0}(x)}{W(x)W(r(x)) \dots W(r^{n_{x_0}(x)-1}(x))}, \quad (x \in \mathcal{T}(x_0)).$$

Moreover, the functions ν_{x_0} are related by

$$(4.17) \quad \nu_{r(x_0)}(x) = W(x_0)\nu_{x_0}(x), \quad (x \in \mathcal{T}(x_0)).$$

Conversely, if ν_{x_0} is an additive function on $\mathcal{T}(x_0)$ for all $x_0 \in X_r$, and the functions satisfy the relation (4.17) then the function h on X_r defined by

$$(4.18) \quad h(x) = \frac{\nu_{r(x)}(x)}{W(x)} = \nu_x(x), \quad (x \in X_r),$$

is R_W -harmonic on X_r .

Proof. Since the restriction of h to $\mathcal{T}(x_0)$ is p -harmonic, the existence and uniqueness of ν_{x_0} such that (4.16) is satisfied follows from Corollary 4.7.

We have for $x \in \mathcal{T}(x_0)$, x is also in $\mathcal{T}(r(x_0))$ so

$$\frac{\nu_{x_0}(x)}{W(x)W(r(x)) \dots W(r^{n_{x_0}(x)-1}(x))} = h(x) = \frac{\nu_{r(x_0)}(x)}{W(x) \dots W(r^{n_{r(x_0)}(x)-1}(x))}.$$

Since $n_{r(x_0)}(x) = n_{x_0}(x) + 1$, and $r^{n_{x_0}(x)}(x) = x_0$, we have

$$W(x_0)\nu_{x_0}(x) = \nu_{r(x_0)}(x).$$

For the converse, we compute

$$\sum_{r(y)=x} W(y)h(y) = \sum_{r(y)=x} W(y) \frac{\nu_{r(y)}(y)}{W(y)} = \sum_{r(y)=x} \nu_x(y) = \nu_x(x) = \frac{\nu_{r(x)}(x)}{W(x)} = h(x).$$

□

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